Note Title 1/13/2010

Improper Integrals

Here's a quick, direct, proof of:

Theorem: Let $f \geq 0$ on $[a, \infty)$, $f \in \mathcal{R}[a,b]$ for all $b > a$. Then $\int_a^\infty f$ converges if and only if $F(b) = \int_a^b f$ is bounded.

<u>Proof</u>: If $\lim_{b\to\infty} \int_a^b f = \lim_{b\to\infty} F(b)$ exists, $F(b)$ must be bounded.

So suppose $S = \{F(b) : b \geq a\}$ is bounded.

Let $L = \sup S$. Then, for every $\epsilon > 0$, there must be a B so that $F(B) > L - \epsilon$. Let $b \geq B$.

Then $\quad L - \epsilon < F(B) \leq F(b) \leq L$. So

$$|F(b) - L| < \epsilon \quad \text{for } b \geq B. \qquad \text{Q.E.D.}$$